On Black-Scholes Equation, Black-Scholes Formula and Binary Option Price

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Abstract:

I. Black-Scholes Equation is derived using two methods: (1) risk-neutral measure; (2) Δ-hedge.
II. The Black-Scholes Formula (the price of European call option is calculated) is calculated using two methods: (1) risk-neutral pricing formula (expected discounted payoff) (2) directly solving the Black-Scholes equation with boundary conditions
III. The two methods in II are proved to be essentially equivalent. The Black-Scholes formula for European call option is tested to be the solution of Black-Scholes equation.
IV. The value of digital options and share digitals are calculated. The European call and put options are be replicated by digital options and share digitals, thus the prices of call and put options can be derived from the values of digitals. The put-call parity relation is given.

1. The derivation(s) of Black-Scholes Equation

Black Scholes model has several assumptions:

1. Constant risk-free interest rate: r
2. Constant drift and volatility of stock price: $dS_t = \mu S_t dt + \sigma S_t dW_t$
3. The stock doesn't pay dividend
4. No arbitrage
5. No transaction fee or cost
6. Possible to borrow and lend any amount (even fractional) of cash at the riskless rate
7. Possible to buy and sell any amount (even fractional) of stock

A typical way to derive the Black-Scholes equation is to claim that under the measure that no arbitrage is allowed (risk-neutral measure), the drift of stock price equal to the risk-free interest rate $\mu = r$. That is (usually under risk-neutral measure, we write Brownian motion as dW_t^Q, here we remove Q subscript for convenience)

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{1}$$

Then apply Ito's lemma to the discounted price of derivatives $e^{-rt}V(t, S_t)$, we get

$$d(e^{-rt}V) = e^{-rt}\left[-rVdt + \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}(dS_t)^2\right]$$
$$= e^{-rt}\left[\left(\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} - rV\right)dt + \sigma S_t\frac{\partial V}{\partial S_t}dW_t\right]$$

(2)

Still, under risk-neutral measure we can argue that $e^{-rt}V(t, S_t)$ is martingale. It should have zero drift. So

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2} = rV$$

(3)

This is the Black-Scholes equation for the price of any derivatives on the underlying S_t, under the Black-Scholes model.

Now I am going to use another method (Δ-hedge method) to derive the Black-Scholes equation. This method avoids to directly use the claim (1).

The price of stock S follows

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

(4)

Apply Ito's lemma to derivative on S:

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_t}dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}(dS_t)^2 = \left(\frac{\partial V}{\partial t} + \mu S_t\frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S_t^2}\right)dt + \sigma S_t\frac{\partial V}{\partial S_t}dW_t$$

(5)

Now construct a portfolio called Δ-hedge portfolio, which shorts one derivative above and holds $\Delta = \frac{\partial V}{\partial S_t}$ shares of stock at time t. The value of this portfolio at time t is

$$\Pi = -V + \frac{\partial V}{\partial S_t}S_t$$

(6)

Before applying Ito's lemma on (6), there is one thing that needs to be emphasized: $\Delta = \frac{\partial V}{\partial S_t}$ is considered to be constant, that is $\frac{\partial \Delta}{\partial t} = 0$ and $\frac{\partial \Delta}{\partial S_t} = 0$, although Δ varies with time t. This result comes from the fact that Δ is determined by S_t at time t and thus should not be considered to be a time-dependent variable when we calculate the change/differential of our portfolio. The differential of our portfolio at time t is

$$d\Pi = -dV + \frac{\partial V}{\partial S_t} dS_t = -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\right) dt \tag{7}$$

The Brownian motion term has vanished! This is a portfolio with riskless return rate of $-\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\right)/\Pi$. Now we use the assumption of no arbitrage, which requires

$$d\Pi = r\Pi dt \tag{8}$$

This yields

$$-\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2}\right) = r\Pi = r\left(-V + \frac{\partial V}{\partial S_t} S_t\right) \tag{9}$$

Here comes the conclusion of Black-Scholes equation

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} = rV \tag{10}$$

Note that (10) doesn't use the risk-neutral measure. Because (10) (or (3)) is a deterministic PDE, it will hold regardless of which measure is used. However, we can see that the use of risk-neutral measure does greatly simplify the derivation.

2. The European vanilla call/put option price

The typical way to derive the European (vanilla) call/put option in many textbook is to calculate the expected discounted payoff of option, in other words to integrate the discounted payoff the risk-neutral measure. This procedure has nothing to do with the Black-Scholes equation we got in (3) or (10). Below I will follow this procedure to get the price $C(t, S_t)$ of a call option on stock S at time t. The call option will mature at time $T > t$ with striking price K.

The price of stock S at time T will be

$$S_T = S_t e^{r\left(T-t+\frac{1}{2}\sigma^2\right)+\sigma W_{T-t}}$$

(11)

The expected discounted payoff of the call option (which is also the price of the call option, from the assumption of no arbitrage) is

$$C(t,S_t) = e^{-r(T-t)}E^Q[\max(S_T - K, 0)] = e^{-r(T-t)}\int_{S_T=K}^{\infty}(S_T - K)dQ = I_1 - I_2$$

(12)

Let $\tau = T - t$. Here $dQ = \frac{1}{\sqrt{2\pi\tau}}e^{-\frac{1}{2\tau}\left(\varepsilon_\tau^Q - \sigma\tau\right)^2}d\varepsilon_\tau^Q$, where $\varepsilon_\tau^Q \sim N(0,\tau)$ is a normal distribution with zero mean and variance $\tau = T - t$. So

$$I_1 = e^{-r(T-t)}\int_{S_T=K}^{\infty} S_T dQ = e^{-r(T-t)}\int_{S_T=K}^{\infty} S_t e^{r\left(T-t+\frac{1}{2}\sigma^2\right)+\sigma\varepsilon_\tau^Q}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{1}{2\tau}\left(\varepsilon_\tau^Q-\sigma\tau\right)^2}d\varepsilon_\tau^Q$$
$$= N(d_1)S_t$$

(13.a)

$$N(d) = \int_{-\infty}^{d}\frac{1}{2\pi}e^{-\frac{x^2}{2}}dx$$

(13.b)

$$d_1 = \frac{\ln\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

(13.c)

$$I_2 = e^{-r(T-t)}\int_{S_T=K}^{\infty} K dQ = e^{-r(T-t)}\int_{S_T=K}^{\infty} K e^{r\left(T-t+\frac{1}{2}\sigma^2\right)+\sigma\varepsilon_\tau^Q}\frac{1}{\sqrt{2\pi\tau}}e^{-\frac{1}{2\tau}\left(\varepsilon_\tau^Q-\sigma\tau\right)^2}d\varepsilon_\tau^Q$$
$$= N(d_2)Ke^{-r(T-t)}$$

(13.d)

$$d_2 = \frac{\ln\frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

(13.e)

So the price of call option at time t is

$$C(t,S_t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

(14)

Equation (14) is also called Black-Scholes formula for vanilla call option, because it can also be derived from Black-Scholes equation (10) with appreciated boundary conditions:

$$C(0,t) = 0 \ \ \forall t$$
$$\lim_{S\to\infty} C(S,t) = S$$
$$C(S,T) = \max(S-K,0)$$

(15.a, 15.b, 15.c)

By the change of variable transformation:

$$\tau = T - t$$
$$u = Ce^{r\tau}$$
$$x = \ln(S) + \left(r - \frac{1}{2}\sigma^2\right)\tau$$

(16.a, 16.b, 16.c)

The Black-Scholes equation (10) becomes the diffusion equation with initial condition

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = u_0(x) \equiv K\left(e^{\max\{x-\ln K, 0\}} - 1\right)$$

(17.a, 17.b)

The solution for diffusion equation is

$$u(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_0[y] \exp\left[-\frac{(x-y)^2}{2\sigma^2\tau}\right] dy$$

(18)

After some math, we have

$$u(x,\tau) = Ke^{x-\ln K+\frac{1}{2}\sigma^2\tau} N(d_1) - KN(d_2)$$

(19)

Changing the variables of $\{u, x, \tau\}$ back to $\{C, S, t\}$ yields the Black-Scholes formula (14).

3. Does the Black-Scholes formula satisfy Black-Scholes equation?

The first method used to derive Black-Scholes formula (14) doesn't use the Black-Scholes equation (10). But it so "happens" to give the solution of Black-Scholes equation (10). This is the "good" property of call/put options: the expected discounted payoff of option is exactly the solution of the Black-Scholes equation.

This property can be extended to other derivatives with different forms of payoffs. For example, if you have a call option on the square of a log-normal asset (like stock price), $C = C(t, S_t^2)$. What equation does the price satisfy? The answer is still Black-Scholes equation, as long as the derivative price is a function of the current time t and stock price S_t. If we derive the price using expected discounted payoff, this price will also satisfy the Black-Scholes equation, i.e. the price from expected discounted payoff is also a solution of Black-Scholes equation.

That is, the solution of Black-Scholes equation for the price any derivative $f(t, S_t)$ is:

$$V(t, S_t) = e^{-r(T-t)} E^Q[V(T, S_T)] \tag{20}$$

The mathematical reason behind this is, $f(t, S_t)$ first of all needs to satisfy the Black-Scholes equation (10):

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S_t^2} = rV$$

We can transform this equation into typical diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2}$$

with change of variables

$$\begin{aligned} \tau &= T - t \\ u &= Ve^{r\tau} \\ x &= \ln(S) + \left(r - \frac{1}{2}\sigma^2\right)\tau \end{aligned}$$

The solution of this diffusion equation with initial boundary condition $u(x, 0) = u_0(x)$ is given in (18):

$$u(x, \tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_0[y] \exp\left[-\frac{(x-y)^2}{2\sigma^2\tau}\right] dy$$

Changing the variables of $\{u, x, \tau\}$ back to $\{C, S, t\}$ yields

$$
\begin{aligned}
V(t,S_t) &= e^{-r(T-t)}\int_{-\infty}^{\infty} V(T,S_T)\frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\left[-\frac{(x-y)^2}{2\sigma^2(T-t)}\right]dy \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty} V(T,S_T)\frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\left[-\frac{y'^2}{2\sigma^2(T-t)}\right]dy' \\
&= e^{-r(T-t)}\int_{-\infty}^{\infty} V(T,S_T)dQ
\end{aligned}
$$

(21)

This is exactly the expected discounted payoff as defined in (20)! So the price of any derivative on S will satisfy Black-Scholes equation, and the solution (Black-Scholes formula) can be calculated from expected discounted payoff (with much easy math).

Now I am going to show in straightforward method that Black-Scholes formula of the price of vanilla call option really satisfies Black-Scholes equation. Recall the price of such call option is

$$C(t,S_t) = N(d_1)S_t + N(d_2)Ke^{-r(T-t)}$$

Define

$$\frac{\partial N(x)}{\partial x} = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \equiv f(x)$$

(22.a)

Then

$$\frac{\partial f(x)}{\partial x} = -xf(x)$$

(22.b)

Also we have

$$
\begin{aligned}
\frac{\partial d_1}{\partial t} &= \frac{d_1}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right) \\
\frac{\partial d_2}{\partial t} &= \frac{d_2}{2(T-t)} - \frac{1}{\sqrt{T-t}}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right) \\
\frac{\partial d_1}{\partial S_t} &= \frac{\partial d_2}{\partial S_t} = \frac{1}{\sigma\sqrt{T-\tau}\,S_t}
\end{aligned}
$$

(22.c, 22.d, 22.e)

Calculate

$$\frac{\partial C}{\partial t} = f(d_1)\frac{\partial d_1}{\partial t}S_t - f(d_2)\frac{\partial d_1}{\partial t}Ke^{-r(T-t)} - rN(d_2)Ke^{-r(T-t)}$$

$$\frac{\partial C}{\partial S_t} = N(d_1) + f(d_1)\frac{\partial d_1}{\partial S_t}S_t - f(d_2)\frac{\partial d_2}{\partial S_t}Ke^{-r(T-t)}$$
$$= N(d_1) + f(d_1)\frac{1}{\sigma\sqrt{T-t}} - f(d_2)Ke^{-r(T-t)}\frac{1}{\sigma\sqrt{T-t}\,S_t}$$
$$\frac{\partial^2 C}{\partial S_t^2} = f(d_1)\frac{\partial d_1}{\partial S_t} - d_1 f(d_1)\frac{\partial d_1}{\partial S_t}\frac{1}{\sigma\sqrt{T-t}} + d_2 f(d_2)\frac{\partial d_2}{\partial S_t}Ke^{-r(T-t)}\frac{1}{\sigma\sqrt{T-t}\,S_t}$$
$$+ f(d_2)Ke^{-r(T-t)}\frac{1}{\sigma\sqrt{T-t}\,S_t^2}$$
$$= \frac{f(d_1)}{\sigma\sqrt{T-\tau}\,S_t} - \frac{d_1 f(d_1)}{\sigma^2(T-t)\,S_t^2} + \frac{d_2 f(d_2)Ke^{-r(T-t)}}{\sigma^2(T-t)\,S_t^2} + \frac{f(d_2)Ke^{-r(T-t)}}{\sigma\sqrt{T-\tau}\;S_t^2}$$

(22.f, 22.g, 22.h)

After some math, we have

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} = rC$$

4. Binary option (also called Digital option)

A binary option pays a fixed amount ($1 for example) in a certain event and zero otherwise. Consider a digital that pays $1at time T if $S(T) > K$. The payoff of such a option is

$$x = \begin{cases} 1 & \text{if } S(T) > K \\ 0 & \text{otherwise} \end{cases}$$

(23)

Using risk-neutral pricing formula

$$C(t, S_t) = e^{-r(T-t)}E^Q[x] = e^{-r(T-t)}prob^Q(S(T) > K) = e^{-r(T-t)}N(d_2)$$

(24)

here $N(d)$ and d_2 are same as defined in (13.b, 13.e).

It is not difficult to check that (24) satisfies Black-Scholes equation (10).

There are 4 kinds of digitals (if we consider dividend $q \neq 0$)

Name	Definition	Value
digit call	Option that pays $1 when $S(T) > K$	$e^{-r(T-t)}N(d_2)$
digit put	Option that pays $1 when $S(T) < K$	$e^{-r(T-t)}N(-d_2)$
share call	Option that pays 1 share when $S(T) > K$	$e^{-q(T-t)}S_tN(d_1)$
share put	Option that pays 1 share when $S(T) < K$	$e^{-q(T-t)}S_tN(-d_1)$

For non-zero dividend, $d_{1,2}$ are slightly different from previous definition (13.c, 13.e)

$$d_1 = \frac{\ln\frac{S_t}{K} + \left(r - q + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{\ln\frac{S_t}{K} + \left(r - q - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

(25.a, 25.b)

With these four digitals, we can easily recover the price of European call and put options. For European call option, use the definition of x in (23), the payoff of this call can be written as

$$C(T, S_T) = xS(T) - xK$$

(26)

This is equivalent to one share call minus K digital call. The combined price of this call option will be

$$C(t, S_t) = e^{-q(T-t)}S_tN(d_1) - Ke^{-r(T-t)}N(d_2)$$

(27)

Similarly, a European put option is equivalent to K digital put minus one share put. The price of the European put option is

$$P(t, S_t) = Ke^{-r(T-t)}N(-d_2) - e^{-q(T-t)}S_tN(-d_1)$$

(28)

The put-call parity is

$$Ke^{-r(T-t)} + C(t, S_t) = e^{-q(T-t)}S_t + P(t, S_t)$$

(29)

This parity follows from the fact that both the left and the right-hand sides are the prices of portfolios that have value $\max(S(T), K)$ at the maturity of the option.

5. Greeks; Δ hedging; Γ hedging